3 - 7 Steady-state solutions

Find the steady-state motion of the mass-spring system modeled by the ODE.

3. y'' + 6 y' + 8 y = 42.5 Cos[2 t]

```
ClearAll["Global`*"]
```

```
hog = y''[t] + 6 y'[t] + 8 y[t] == 42.5 Cos[2 t]
nar = DSolve[hog, y[t], t]
```

```
8 y[t] + 6 y′[t] + y′′[t] == 42.5 Cos[2 t]
```

$$\{\{y[t] \to e^{-4. t} C[1] + e^{-2. t} C[2] + 1.0625 (1. Cos[2. t] + 3. Sin[2. t])\}\}$$

```
Expand[
 1.0624999999999991` (1.` Cos[2.` t] + 3.000000000000002` Sin[2.` t])]
```

```
1.0625 Cos[2. t] + 3.1875 Sin[2. t]
```

1. Above: one section of ‘nar’ is expanded ‘by hand’.

```
nar /.
 (1.0624999999999991` (1.` Cos[2.` t] + 3.000000000000002` Sin[2.` t])) ->
  1.0624999999999991` Cos[2.` t] + 3.1874999999999996` Sin[2.` t]
```

$$\{\{y[t] \to e^{-4. t} C[1] + e^{-2. t} C[2] + 1.0625 Cos[2. t] + 3.1875 Sin[2. t]\}\}$$

2. Above: expanded section reinserted into ‘nar’. This version matches the text answer, if the two constant coefficients C[1] and C[2] assume a value of zero.

5. $(D^2 + D + 4.25 I)\ y = 22.1 Cos[4.5]\ t$

```
ClearAll["Global`*"]
```

```
opa = y''[t] + y'[t] + 4.25 y[t] == 22.1 Cos[4.5 t]
erb = DSolve[opa, y[t], t]
```

```
4.25 y[t] + y′[t] + y′′[t] == 22.1 Cos[4.5 t]
```

$$\{\{y[t] \to e^{-0.5 t} C[2] Cos[2. t] + e^{-0.5 t} C[1] Sin[2. t] - 2.125 (1. Cos[2. t] Cos[2.5 t] - 0.397647 Cos[2. t] Cos[6.5 t] - 0.2 Cos[2.5 t] Sin[2. t] - 0.0305882 Cos[6.5 t] Sin[2. t] - 0.2 Cos[2. t] Sin[2.5 t] - 1. Sin[2. t] Sin[2.5 t] + 0.0305882 Cos[2. t] Sin[6.5 t] - 0.397647 Sin[2. t] Sin[6.5 t])\}\}$$

```
latch = TrigReduce[erb]
```

$$\{\{y[t] \to -1.28 e^{-0.5 t} (-0.78125 C[2] Cos[2. t] + 1. e^{0.5 t} Cos[4.5 t] + 4.60786 \times 10^{-17} e^{0.5 t} Cos[8.5 t] - 0.78125 C[1] Sin[2. t] - 0.28125 e^{0.5 t} Sin[4.5 t])\}\}$$

```
Simplify[latch]
```

$\{\{y[t] \to 1.\, e^{-0.5\,t}\, C[2]\, \text{Cos}[2.\, t] - 1.28\, \text{Cos}[4.5\, t] - 5.89806 \times 10^{-17}\, \text{Cos}[8.5\, t] + 1.\, e^{-0.5\,t}\, C[1]\, \text{Sin}[2.\, t] + 0.36\, \text{Sin}[4.5\, t]\}\}$

```
redondo = Simplify[Chop[latch, 10^-16]]
```

$\{\{y[t] \to 1.\, e^{-0.5\,t}\, C[2]\, \text{Cos}[2.\, t] - 1.28\, \text{Cos}[4.5\, t] + 1.\, e^{-0.5\,t}\, C[1]\, \text{Sin}[2.\, t] + 0.36\, \text{Sin}[4.5\, t]\}\}$

```
narv = Simplify[redondo /. {C[1] → 0, C[2] → 0}]
```

$\{\{y[t] \to -1.28\, \text{Cos}[4.5\, t] + 0.36\, \text{Sin}[4.5\, t]\}\}$

1. The above result matches the text answer.

7. $(4\, D^2 + 12\, D + 9\, I)\, y = 225 - 75\, \text{Sin}[3\, t]$

```
ClearAll["Global`*"]
```

```
halli = 4 y''[t] + 12 y'[t] + 9 y[t] == 225 - 75 Sin[3 t]
wan = DSolve[halli, y[t], t]
```

$9\, y[t] + 12\, y'[t] + 4\, y''[t] == 225 - 75\, \text{Sin}[3\, t]$

$\{\{y[t] \to e^{-3\,t/2}\, C[1] + e^{-3\,t/2}\, t\, C[2] + \frac{1}{3}\, (75 + 4\, \text{Cos}[3\, t] + 3\, \text{Sin}[3\, t])\}\}$

```
Simplify[wan]
```

$\{\{y[t] \to 25 + e^{-3\,t/2}\, C[1] + e^{-3\,t/2}\, t\, C[2] + \frac{4}{3}\, \text{Cos}[3\, t] + \text{Sin}[3\, t]\}\}$

```
sz = Simplify[wan /. {C[1] → 0, C[2] → 0}]
```

$\{\{y[t] \to 25 + \frac{4}{3}\, \text{Cos}[3\, t] + \text{Sin}[3\, t]\}\}$

1. The above result matches the text answer.

8 - 15 Transient solutions

Find the transient motion of the mass-spring system modeled by the ODE.

9. y'' + 3 y' + 3.25 y = 3 Cos[t] - 1.5 Sin[t]

```
ClearAll["Global`*"]
```

I ran across a perfect way to do the method of undetermined coefficients in *Mathematica*, for problems like this, at *https://mathematica.stackexchange.com/questions/159382/using-the-method-of-undetermined-coefficients*, response of Nasser.

```
(*returns homogeneous and particular solutions*)
hAndp[odeH_, rhs_, y_, x_] :=
  Module[{wronskian, u1, u2, solH, y1, y2, leadingC},
   leadingC = Cases[odeH, c_ y''[x] :→ c];
   leadingC = If[leadingC === {}, 1, First@leadingC];
   solH = (y[x] /. First@DSolve[odeH == 0, y[x], x]);
   {y1, y2} = solH /. C[1] y1_ + C[2] y2_ :→ {y1, y2};
   (*basis solutions*)
   wronskian = Det[{{y1, y2}, {D[y1, x], D[y2, x]}}];
   u1 = -Integrate[y2 rhs / (leadingC * wronskian), x];
   u2 = Integrate[y1 rhs / (leadingC * wronskian), x];
   {solH, Simplify[y1 u1 + y2 u2]}];

odeH = y''[t] + 3. y'[t] + 3.25 y[t];
rhs = 3. Cos[t] - 1.5 Sin[t];

{yh, yp} = hAndp[odeH, rhs, y, t]

{ⅇ^(-1.5 t) C[2] Cos[1. t] + ⅇ^(-1.5 t) C[1] Sin[1. t],
 0.3 Cos[t] + 0.5 Cos[(1. + 0. ⅈ) t] - 0.6 Sin[t] + 1. Sin[(1. + 0. ⅈ) t]}

fullSolution = yh + yp

0.3 Cos[t] + ⅇ^(-1.5 t) C[2] Cos[1. t] + 0.5 Cos[(1. + 0. ⅈ) t] -
 0.6 Sin[t] + ⅇ^(-1.5 t) C[1] Sin[1. t] + 1. Sin[(1. + 0. ⅈ) t]

colsol = Collect[fullSolution, ⅇ^(-1.5 t)]

0.3 Cos[t] + 0.5 Cos[(1. + 0. ⅈ) t] - 0.6 Sin[t] +
 ⅇ^(-1.5 t) (C[2] Cos[1. t] + C[1] Sin[1. t]) + 1. Sin[(1. + 0. ⅈ) t]

colsolSH = 0.8 Cos[t] + 0.4 Sin[t] + ⅇ^(-1.5 t) (C[2] Cos[t] + C[1] Sin[t])

0.8 Cos[t] + 0.4 Sin[t] + ⅇ^(-1.5 t) (C[2] Cos[t] + C[1] Sin[t])
```

The solution in green above matches the answer in the text. However, I have not been successful so far in checking the answer through differentiation and substitution. Whatever problem the following crude attempt has, it is a big one.

```
cls[t_] = 0.8 Cos[t] + 0.4 Sin[t] + ⅇ^(-1.5 t) (Cos[t] + Sin[t])
0.8 Cos[t] + 0.4 Sin[t] + ⅇ^(-1.5 t) (Cos[t] + Sin[t])

cd[t_] = D[cls, t];

cd2[t_] = D[cls, {t, 2}];
```

```
Grid[Table[{cd2[k] + 3 cd[k] + 3.25 cls[k], 3 Cos[k] - 1.5 Sin[k]},
  {k, {1, 2, ⅇ, 3, π}}], Frame → All]
```

3.50072	0.3587
0.179901	-2.61239
-1.86409	-3.35137
-2.42117	-3.18166
-2.6292	-3.

11. $(D^2 + 2\,I)\,y = \text{Cos}\left[\sqrt{2}\,t\right] + \text{Sin}\left[\sqrt{2}\,t\right]$

```
eqn = y''[x] - (a*x^6 + x^2)*y[x];
sol = DSolve[eqn == 0, y, x]
```

```
ClearAll["Global`*"]
```

I reworked Nasser's module with t instead of x, in the hope it would show the reason for the difficulty.

```
(*returns homogeneous and particular solutions*)
hAndp[odeH_, rhs_, y_, t_] :=
  Module[{wronskian, u1, u2, solH, y1, y2, leadingC},
   leadingC = Cases[odeH, c_ y''[t] :→ c];
   leadingC = If[leadingC === {}, 1, First@leadingC];
   solH = (y[t] /. First@DSolve[odeH == 0, y[t], t]);
   {y1, y2} = solH /. C[1] y1_ + C[2] y2_ :→ {y1, y2};
   (*basis solutions*)
   wronskian = Det[{{y1, y2}, {D[y1, t], D[y2, t]}}];
   u1 = -Integrate[y2 rhs / (leadingC * wronskian), t];
   u2 = Integrate[y1 rhs / (leadingC * wronskian), t];
   {solH, Simplify[y1 u1 + y2 u2]}];
```

```
odeH = y''[t] + 2 y[t];
rhs = Cos[√2 t] + Sin[√2 t];
```

The module still performs.

```
{yh, yp} = hAndp[odeH, rhs, y, t]
```

$$\left\{C[1]\,\text{Cos}\left[\sqrt{2}\,t\right] + C[2]\,\text{Sin}\left[\sqrt{2}\,t\right],\ \frac{\left(\sqrt{2} - 4\,t\right)\text{Cos}\left[\sqrt{2}\,t\right] + \left(\sqrt{2} + 4\,t\right)\text{Sin}\left[\sqrt{2}\,t\right]}{8\,\sqrt{2}}\right\}$$

The module comes up with a solution which looks a little like the text answer, but not quite.

```
fullSolution = yh + yp
```

$$C[1]\,\mathrm{Cos}[\sqrt{2}\,t] + C[2]\,\mathrm{Sin}[\sqrt{2}\,t] + \frac{(\sqrt{2} - 4\,t)\,\mathrm{Cos}[\sqrt{2}\,t] + (\sqrt{2} + 4\,t)\,\mathrm{Sin}[\sqrt{2}\,t]}{8\sqrt{2}}$$

Mathematica doubles down on the suspicious solution by DSolving it directly. This is a forward test, not a back test.

```
y[t] /. First@DSolve[odeH - rhs == 0, y[t], t]
```

$$C[1]\,\mathrm{Cos}[\sqrt{2}\,t] + C[2]\,\mathrm{Sin}[\sqrt{2}\,t] + \frac{1}{8\sqrt{2}}\Big(-4\,t\,\mathrm{Cos}[\sqrt{2}\,t] + \sqrt{2}\,\mathrm{Cos}[\sqrt{2}\,t]\,\mathrm{Cos}[2\sqrt{2}\,t] + 4\,t\,\mathrm{Sin}[\sqrt{2}\,t] - \sqrt{2}\,\mathrm{Cos}[2\sqrt{2}\,t]\,\mathrm{Sin}[\sqrt{2}\,t] + \sqrt{2}\,\mathrm{Cos}[\sqrt{2}\,t]\,\mathrm{Sin}[2\sqrt{2}\,t] + \sqrt{2}\,\mathrm{Sin}[\sqrt{2}\,t]\,\mathrm{Sin}[2\sqrt{2}\,t]\Big)$$

Cutting out the latter part, the ‘tail’, of the proposed solution, which seems to contain the wayward-looking content.

$$\mathrm{outy} = \frac{1}{8\sqrt{2}}\Big(-4\,t\,\mathrm{Cos}[\sqrt{2}\,t] + \sqrt{2}\,\mathrm{Cos}[\sqrt{2}\,t]\,\mathrm{Cos}[2\sqrt{2}\,t] + 4\,t\,\mathrm{Sin}[\sqrt{2}\,t] - \sqrt{2}\,\mathrm{Cos}[2\sqrt{2}\,t]\,\mathrm{Sin}[\sqrt{2}\,t] + \sqrt{2}\,\mathrm{Cos}[\sqrt{2}\,t]\,\mathrm{Sin}[2\sqrt{2}\,t] + \sqrt{2}\,\mathrm{Sin}[\sqrt{2}\,t]\,\mathrm{Sin}[2\sqrt{2}\,t]\Big);$$

The tail is tested on an integer value.

```
N[outy /. t → 2]
0.810144
```

$$N\Big[\frac{(\sqrt{2} - 4\,t)\,\mathrm{Cos}[\sqrt{2}\,t] + (\sqrt{2} + 4\,t)\,\mathrm{Sin}[\sqrt{2}\,t]}{8\sqrt{2}} \;/.\; t \to 2\Big]$$

The other version of the tail is also tested.

```
0.810144
```

$$N\Big[\frac{t\,(\mathrm{Sin}[\sqrt{2}\,t] - \mathrm{Cos}[\sqrt{2}\,t])}{2\sqrt{2}} \;/.\; t \to 2\Big]$$

The text answer ‘tail’ comes back with a different value.

```
0.890555
```

13. $(D^2 + I)\,y = \mathrm{Cos}[\omega\,t],\ \omega^2 \neq 1$

```
ClearAll["Global`*"]
```

```
(*returns homogeneous and particular solutions*)
hAndp[odeH_, rhs_, y_, x_] :=
  Module[{wronskian, u1, u2, solH, y1, y2, leadingC},
   leadingC = Cases[odeH, c_ y''[x] :→ c];
   leadingC = If[leadingC === {}, 1, First@leadingC];
   solH = (y[x] /. First@DSolve[odeH == 0, y[x], x]);
   {y1, y2} = solH /. C[1] y1_ + C[2] y2_ :→ {y1, y2};
   (*basis solutions*)
   wronskian = Det[{{y1, y2}, {D[y1, x], D[y2, x]}}];
   u1 = -Integrate[y2 rhs / (leadingC * wronskian), x];
   u2 = Integrate[y1 rhs / (leadingC * wronskian), x];
   {solH, Simplify[y1 u1 + y2 u2]}];
```

Oddly, the Mathematica machine objected when the symbol ω was used, but not when the symbol a was used. I couldn't get accommodation for the Assumptions on a, but it didn't seem to gum up the works.

```
odeH = y''[t] + y[t];
rhs = Cos[a t];
```

```
{yh, yp} = hAndp[odeH, rhs, y, t]
```

$$\left\{C[1] \operatorname{Cos}[t] + C[2] \operatorname{Sin}[t], \frac{\operatorname{Cos}[a\, t]}{1 - a^2}\right\}$$

The sum of the two parts equals the text answer.

```
fullSolution = yh + yp
```

$$C[1] \operatorname{Cos}[t] + \frac{\operatorname{Cos}[a\, t]}{1 - a^2} + C[2] \operatorname{Sin}[t]$$

A forward checking step is available.

```
y[t] /. First@DSolve[odeH - rhs == 0, y[t], t]
```

$$C[1] \operatorname{Cos}[t] + C[2] \operatorname{Sin}[t] + \frac{-\operatorname{Cos}[t]^2 \operatorname{Cos}[a\, t] - \operatorname{Cos}[a\, t] \operatorname{Sin}[t]^2}{-1 + a^2}$$

15. $\left(D^2 + 4 D + 8 I\right) y = 2 \operatorname{Cos}[2 t] + \operatorname{Sin}[2 t]$

```
ClearAll["Global`*"]
```

```
(*returns homogeneous and particular solutions*)
hAndp[odeH_, rhs_, y_, x_] :=
  Module[{wronskian, u1, u2, solH, y1, y2, leadingC},
   leadingC = Cases[odeH, c_ y''[x] :> c];
   leadingC = If[leadingC === {}, 1, First@leadingC];
   solH = (y[x] /. First@DSolve[odeH == 0, y[x], x]);
   {y1, y2} = solH /. C[1] y1_ + C[2] y2_ :> {y1, y2};
   (*basis solutions*)
   wronskian = Det[{{y1, y2}, {D[y1, x], D[y2, x]}}];
   u1 = -Integrate[y2 rhs / (leadingC * wronskian), x];
   u2 = Integrate[y1 rhs / (leadingC * wronskian), x];
   {solH, Simplify[y1 u1 + y2 u2]}];
```

No special circumstances this time. The module runs smoothly.

```
odeH = y''[t] + 4 y'[t] + 8 y[t];
rhs = 2 Cos[2 t] + Sin[2 t];
{yh, yp} = hAndp[odeH, rhs, y, t]
```

$\left\{e^{-2t} C[2] \operatorname{Cos}[2t] + e^{-2t} C[1] \operatorname{Sin}[2t], \frac{1}{4} \operatorname{Sin}[2t]\right\}$

The sum of the two parts matches the answer in the text.

```
fullSolution = yh + yp
```

$e^{-2t} C[2] \operatorname{Cos}[2t] + \frac{1}{4} \operatorname{Sin}[2t] + e^{-2t} C[1] \operatorname{Sin}[2t]$

The forward check shows agreement with the module output.

```
y[t] /. First@DSolve[odeH - rhs == 0, y[t], t]
```

$e^{-2t} C[2] \operatorname{Cos}[2t] + e^{-2t} C[1] \operatorname{Sin}[2t] + \frac{1}{8}\left(-8 \operatorname{Cos}[t]^2 \operatorname{Cos}[2t] \operatorname{Sin}[t]^2 + 2 \operatorname{Sin}[2t] + \operatorname{Sin}[2t] \operatorname{Sin}[4t]\right)$ // FullSimplify

$\frac{1}{4} \operatorname{Sin}[2t] + e^{-2t} (C[2] \operatorname{Cos}[2t] + C[1] \operatorname{Sin}[2t])$

So out of four test problems, the Nasser module performs acceptably on three. A welcome method for those undetermined coefficient situations.

16 - 20 Initial value problems

Find the motion of the mass-spring system modeled by the ODE and the initial conditions. Sketch or graph the solution curve. In addition, sketch or graph the curve of y - y_p to see when the system practically reaches the steady state.

17. $\left(D^2 + 4\,I\right) y = \text{Sin}[t] + \frac{1}{3}\text{Sin}[3\,t] + \frac{1}{5}\text{Sin}[5\,t],\ y[0] = 0,\ y'[0] = \frac{3}{35}$

```
ClearAll["Global`*"]
```

```
jolt =
 {y''[t] + 4 y[t] == Sin[t] + 1/3 Sin[3 t] + 1/5 Sin[5 t], y[0] == 0, y'[0] == 3/35}
holt = DSolve[jolt, y[t], t]
```

$$\left\{4\,y[t] + y''[t] == \text{Sin}[t] + \frac{1}{3}\text{Sin}[3\,t] + \frac{1}{5}\text{Sin}[5\,t],\ y[0] == 0,\ y'[0] == \frac{3}{35}\right\}$$

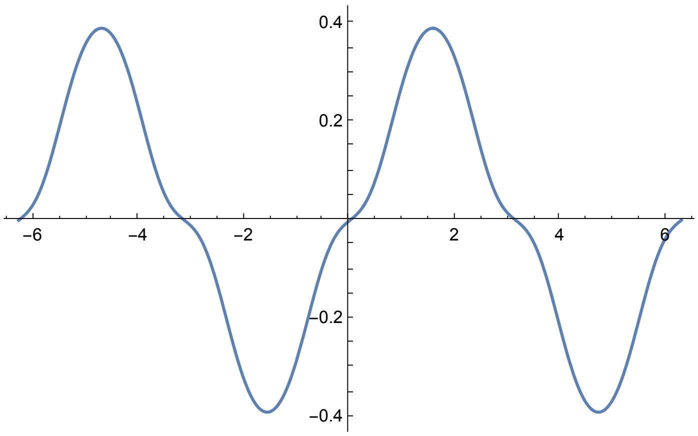
$$\left\{\left\{y[t] \to \frac{1}{1260}\left(63\,\text{Cos}[2\,t]\,\text{Sin}[t] - 644\,\text{Cos}[2\,t]\,\text{Sin}[t]^3 + 210\,\text{Cos}[t]\,\text{Sin}[2\,t] - 126\,\text{Cos}[3\,t]\,\text{Sin}[2\,t] - 21\,\text{Cos}[5\,t]\,\text{Sin}[2\,t] - 9\,\text{Cos}[7\,t]\,\text{Sin}[2\,t] - 77\,\text{Cos}[2\,t]\,\text{Sin}[3\,t] + 21\,\text{Cos}[2\,t]\,\text{Sin}[5\,t] + 9\,\text{Cos}[2\,t]\,\text{Sin}[7\,t]\right)\right\}\right\}$$

```
TrigReduce[holt]
```

$$\left\{\left\{y[t] \to \frac{1}{105}\left(35\,\text{Sin}[t] - 7\,\text{Sin}[3\,t] - \text{Sin}[5\,t]\right)\right\}\right\}$$

1. Above: This expression matches the text answer.

```
Plot[y[t] /. holt, {t, -2 π, 2 π}, PlotRange → Automatic]
```



19. $\left(D^2 + 2\,D + 2\,I\right) y = e^{-t/2}\,\text{Sin}\left[\frac{1}{2}\,t\right],\ y[0] = 0,\ y'[0] = 1$

```
ClearAll["Global`*"]
```

```
num = {y''[t] + 2 y'[t] + 2 y[t] == e^(-t/2) Sin[1/2 t], y[0] == 0, y'[0] == 1}
stum = DSolve[num, y[t], t]
```

$$\left\{2\,y[t] + 2\,y'[t] + y''[t] == e^{-t/2}\,\mathrm{Sin}\left[\frac{t}{2}\right],\ y[0] == 0,\ y'[0] == 1\right\}$$

$$\left\{\left\{y[t] \to -\frac{1}{10} e^{-t} \left(-4\,\mathrm{Cos}[t] + 5\,e^{t/2}\,\mathrm{Cos}\left[\frac{t}{2}\right]\mathrm{Cos}[t] - e^{t/2}\,\mathrm{Cos}[t]\,\mathrm{Cos}\left[\frac{3t}{2}\right] + 5\,e^{t/2}\,\mathrm{Cos}[t]\,\mathrm{Sin}\left[\frac{t}{2}\right] - 8\,\mathrm{Sin}[t] - 5\,e^{t/2}\,\mathrm{Cos}\left[\frac{t}{2}\right]\mathrm{Sin}[t] + 3\,e^{t/2}\,\mathrm{Cos}\left[\frac{3t}{2}\right]\mathrm{Sin}[t] + 5\,e^{t/2}\,\mathrm{Sin}\left[\frac{t}{2}\right]\mathrm{Sin}[t] - 3\,e^{t/2}\,\mathrm{Cos}[t]\,\mathrm{Sin}\left[\frac{3t}{2}\right] - e^{t/2}\,\mathrm{Sin}[t]\,\mathrm{Sin}\left[\frac{3t}{2}\right]\right)\right\}\right\}$$

```
blum = TrigReduce[stum]
```

$$\left\{\left\{y[t] \to -\frac{2}{5} e^{-t} \left(e^{t/2}\,\mathrm{Cos}\left[\frac{t}{2}\right] - \mathrm{Cos}[t] - 2\,e^{t/2}\,\mathrm{Sin}\left[\frac{t}{2}\right] - 2\,\mathrm{Sin}[t]\right)\right\}\right\}$$

```
rum = Collect[blum, e^(t/2)]
```

$$\left\{\left\{y[t] \to -\frac{2}{5} e^{-t/2} \left(\mathrm{Cos}\left[\frac{t}{2}\right] - 2\,\mathrm{Sin}\left[\frac{t}{2}\right]\right) - \frac{2}{5} e^{-t} \left(-\mathrm{Cos}[t] - 2\,\mathrm{Sin}[t]\right)\right\}\right\}$$

1. Above: Marking the first time I used **Collect** that it worked fairly well.

```
tum = rum /. (-2/5 (Cos[t/2] - 2 Sin[t/2])) → (-2/5 Cos[t/2] + 4/5 Sin[t/2])
```

$$\left\{\left\{y[t] \to e^{-t/2} \left(-\frac{2}{5}\,\mathrm{Cos}\left[\frac{t}{2}\right] + \frac{4}{5}\,\mathrm{Sin}\left[\frac{t}{2}\right]\right) - \frac{2}{5} e^{-t} \left(-\mathrm{Cos}[t] - 2\,\mathrm{Sin}[t]\right)\right\}\right\}$$

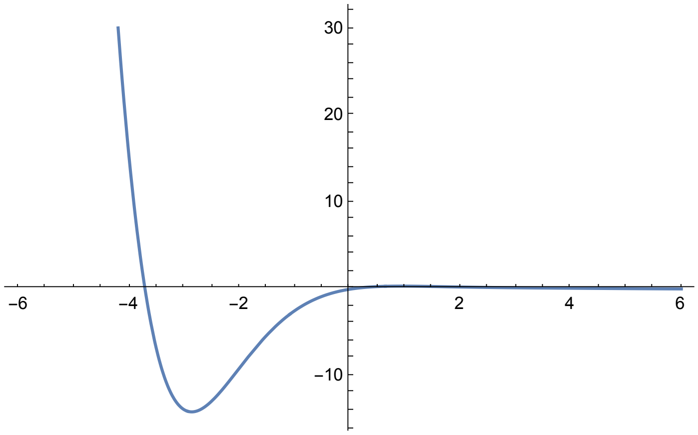
2. Above: I feel confident that using **Expand** would have messed it up, so I did the distribution by hand. I did only one half of it, leaving the other half for the next step.

```
zum = tum /. (-2/5 e^-t (-Cos[t] - 2 Sin[t])) → (e^-t (2/5 Cos[t] + 4/5 Sin[t]))
```

$$\left\{\left\{y[t] \to e^{-t/2} \left(-\frac{2}{5}\,\mathrm{Cos}\left[\frac{t}{2}\right] + \frac{4}{5}\,\mathrm{Sin}\left[\frac{t}{2}\right]\right) + e^{-t} \left(\frac{2\,\mathrm{Cos}[t]}{5} + \frac{4\,\mathrm{Sin}[t]}{5}\right)\right\}\right\}$$

3. Above: Carrying out the other half of the distribution of constants inside parentheses. At this point the answer matches the text answer, except that rationals are retained in fractional form.

```
Plot[y[t] /. zum, {t, -6, 6}, PlotRange -> Automatic]
```



21. Beats. Derive the formula after (12) from (12). Can we have beats in a damped system?

23. Team experiment. Practical resonance.
(a) Derive, in detail, the crucial formula (16).
(b) By considering $\frac{dC*}{dc}$ show that $C*(\omega_{max})$ increases as c ($\leq \sqrt{2\,mk}$ decreases.
(c) Illustrate practical resonance with an ODE of your own in which you vary c, and sketch or graph corresponding curves as in fig 57.
(d) Take your ODE with c fixed and an input of two terms, one with frequency and the other not. Discuss and sketch or graph the output.
(e) Give other applications (not in the book) in which resonance is important.

```
ClearAll["Global`*"]
```

After playing with it awhile, I can't make it look anything like figure 57.

```
Table[Plot[2/(c Sqrt[4 ω^2 - c^2]), {ω, 0, 2},
   PlotRange → {{0, 2}, {-2, 12}}], {c, 0.1, 2, 0.4}];
```

However I did run across the exact desired plot on the site of Nasser M. Abbasi, *https://12000.org/my_notes/mma_matlab_control/KERNEL2/index.htm#x1-20001.1*, at approx 22 percent scroll. Text notes from that site include the following: "Problem: Plot the standard curves showing how the dynamic response R_d changes as r = $\frac{\omega}{\omega_n}$ changes. Do this for different damping ratio ξ. Also plot the phase angle. These plots are the result of analysis of the response of a second order damped system to a harmonic loading. ω is the forcing frequency and ω_n is the natural frequency of the system."

Note: I have not yet included the plot of the phase angles.

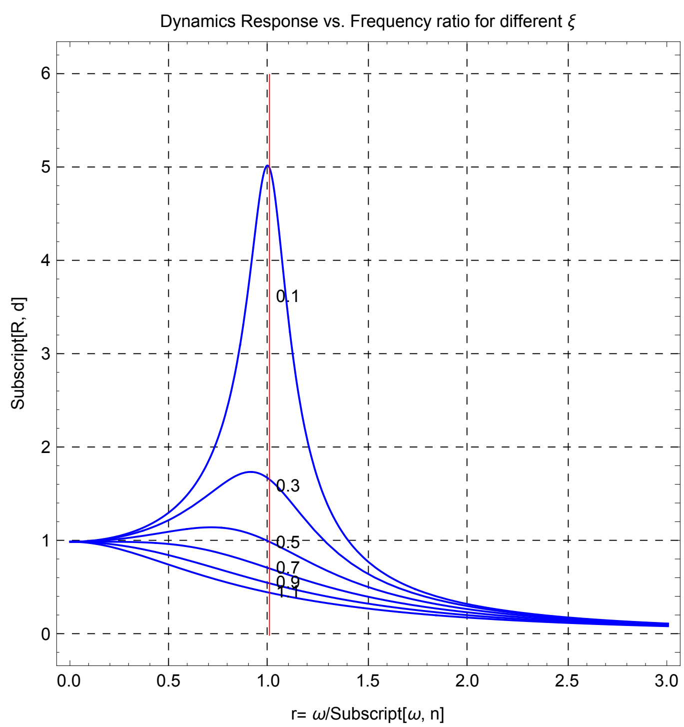
```
Rd[r_, z_] := 1/Sqrt[(1 - r^2)^2 + (2 z r)^2];

phase[r_, z_] := Module[{t}, t = ArcTan[(2 z r)/(1 - r^2)];
   If[t < 0, t = t + Pi];
   180/Pi t];

plotOneZeta[z_, f_] :=
  Module[{r, p1, p2}, p1 = Plot[f[r, z], {r, 0, 3}, PlotRange → All,
     PlotStyle → {Blue, Thickness[0.003]}];
   p2 = Graphics[Text[z, {1.1, 1.1 f[1.1, z]}]];
   Show[{p1, p2}]];

p1 = Graphics[{Red, Line[{{1, 0}, {1, 6}}]}];
p2 = Map[plotOneZeta[#, Rd] &, Range[.1, 1.2, .2]];

Show[p2, p1,
 FrameLabel → {{"Subscript[R, d]", None}, {"r= ω/Subscript[ω, n]",
    "Dynamics Response vs. Frequency ratio for different ξ"}},
 Frame → True, GridLines → Automatic, GridLinesStyle → Dashed,
 ImageSize → 350, AspectRatio → 1]
```



25. CAS Experiment. Undamped vibrations.

(a) Solve the initial value problem

$y'' + y = \cos(\omega t)$, $\omega^2 \neq 1$, $y[0] = 0$, $y'[0] = 0$.

Show that the solution can be written

$y[t] = \frac{2}{1-\omega^2} \sin\left[\frac{1}{2}(1+\omega)\, t\right] \sin\left[\frac{1}{2}(1-\omega)\, t\right]$.

(b) Experiment with the solution by changing ω to see the change of the curves from those for small ω (>0) to beats, to resonance, and to large values of ω (see the figure below).

```
ClearAll["Global`*"]
```

Part (a). With the green cell below showing true, part (a) is complete.

```
eqn = y''[t] + y[t] == Cos[ω t]
y[t] + y''[t] == Cos[t ω]
```

```
sol = DSolve[{eqn, y[0] == 0, y'[0] == 0}, y, t, Assumptions → ω^2 ≠ 1]
```

$$\left\{\left\{y \to \text{Function}\left[\{t\}, \frac{\text{Cos}[t] - \text{Cos}[t]^2\,\text{Cos}[t\,\omega] - \text{Cos}[t\,\omega]\,\text{Sin}[t]^2}{-1 + \omega^2}\right]\right\}\right\}$$

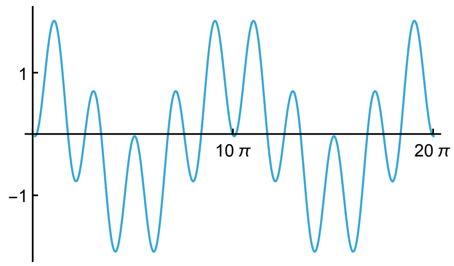
$$\text{PossibleZeroQ}\left[\frac{2\,\text{Sin}\left[\frac{1}{2}(1+\omega)\,t\right]\,\text{Sin}\left[\frac{1}{2}(1-\omega)\,t\right]}{1-\omega^2} - \frac{\text{Cos}[t] - \text{Cos}[t]^2\,\text{Cos}[t\,\omega] - \text{Cos}[t\,\omega]\,\text{Sin}[t]^2}{-1+\omega^2}\right]$$

```
True
```

Part (b). Many possible versions of the solution function can be had. The following three, which resemble the plots in figure 60 in the text, are a good sample.

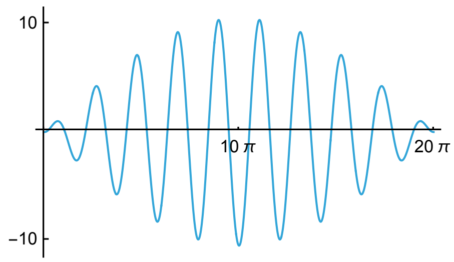
$$\text{Labeled}\left[\text{Plot}\left[\text{Table}\left[\frac{2\,\text{Sin}\left[\frac{1}{2}(1+\omega)\,t\right]\,\text{Sin}\left[\frac{1}{2}(1-\omega)\,t\right]}{1-\omega^2}, \{\omega, \{0.2\}\}\right],\right.\right.$$

```
  {t, 0, 20 π}, PlotStyle → {Thickness[0.005], RGBColor[0.2, 0.65, 0.85]},
  Ticks → {{0, 10 Pi, 20 Pi}, {-1, 1}},
  AxesStyle → Thickness[0.004], ImageSize → 230], "ω=0.2"]
```



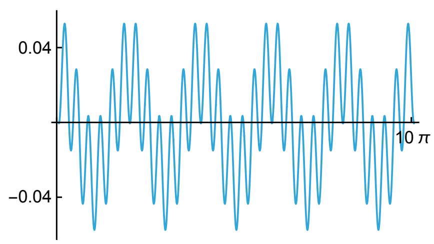
ω=0.2

```
Labeled[Plot[Table[(2 Sin[1/2 (1 + ω) t] Sin[1/2 (1 - ω) t])/(1 - ω^2), {ω, {0.9}}],
  {t, 0, 20 π}, PlotStyle → {Thickness[0.005], RGBColor[0.2, 0.65, 0.85]},
  Ticks → {{0, 10 Pi, 20 Pi}, {-10, 10}},
  AxesStyle → Thickness[0.004], ImageSize → 230], "ω=0.9"]
```



ω=0.9

```
Labeled[Plot[Table[(2 Sin[1/2 (1 + ω) t] Sin[1/2 (1 - ω) t])/(1 - ω^2), {ω, {6}}],
  {t, 0, 10 π}, PlotStyle → {Thickness[0.005], RGBColor[0.2, 0.65, 0.85]},
  Ticks → {{0, 10 Pi, 20 Pi}, {-0.04, 0.04}},
  AxesStyle → Thickness[0.004], ImageSize → 220], "ω=6"]
```



ω=6